Formelsammlung

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October 7, 2024

Contents

linalg I

Linear algebra

1 Determinant

2x2 matrix	$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,d - c\,b \quad (1)$$
3x3 matrix (Rule of Sarrus)	$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\,e\,i + b\,f\,g + c\,d\,h - g\,e\,c - h\,f\,a - i\,d\,b \quad (2)$$
Leibniz formla	$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right) \quad (3)$$
Product	$$\det(A\,B) = \det(A)\det(B) \quad (4)$$
Inverse	$$\det(A^{-1}) = \det(A)^{-1} \quad (5)$$
Transposed	$$\det(A^{\mathrm{T}}) = \det(A) \quad (6)$$

2 linalg:zeug

Inverse 2×2 matrix	$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (7)$$
Unitary matrix	$$U^{\dagger} U = \mathbb{1} \quad (8)$$

Singular value decomposition Factorization of complex matrices through rotating →rescaling →rotation.	$$A = U\Lambda V \quad (9)$$ A: $m \times n$ matrix, U: $m \times m$ unitary matrix, Λ: $m \times n$ rectangular diagonal matrix with non-negative numbers on the diagonal, V: $n \times n$ unitary matrix
2D rotation matrix	$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad (10)$$
3D rotation matrices	$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad (11)$$ $$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad (12)$$ $$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (13)$$
Properites of rotation matrices	$$R^{\mathrm{T}} = R^{-1} \quad (14)$$ $$\det R = 1 \quad (15)$$ $$R \in \mathrm{SO}(n) \quad (16)$$ n dimension, $\mathrm{SO}(n)$ special othogonal group

3 Eigenvalues

Eigenvalue equation	$$Av = \lambda v \quad (17)$$ λ eigenvalue, v eigenvector
Characteristic polynomial Zeros are the eigenvalues of A	$$\chi_A = \det(A - \lambda \mathbb{1}) \stackrel{!}{=} 0 \quad (18)$$
Kramer's theorem If H is invariant under T and $\|\psi\rangle$ is an eigenstate of H, then $T\|\psi\rangle$ is also am eigenstate of H	$$THT^\dagger = H \quad \wedge \quad H\|\psi\rangle = E\|\psi\rangle \quad \Rightarrow \quad HT\|\psi\rangle = ET\|\psi\rangle \quad (19)$$

Eigendecomposition	$$A = V\Lambda V^{-1} \quad (20)$$ A diagonalizable, columns of V are eigenvectors v_i, Λ diagonal matrix with eigenvalues λ_i on the diagonal

TODO:Jordan stuff, blockdiagonal matrices, permutations, skalar product lapacescher entwicklungssatz maybe, cramers rule geo II

Geometry

4 Trigonometry

Exponential function	$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{21}$$
Sine	$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!} \tag{22}$$ $$= \frac{e^{ix} - e^{-ix}}{2i} \tag{23}$$
Cosine	$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n)}}{(2n)!} \tag{24}$$ $$= \frac{e^{ix} + e^{-ix}}{2} \tag{25}$$
Hyperbolic sine	$$\sinh(x) = -i \sin ix \tag{26}$$ $$= \frac{e^{x} - e^{-x}}{2} \tag{27}$$
Hyperbolic cosine	$$\cosh(x) = \cos ix \tag{28}$$ $$= \frac{e^{x} + e^{-x}}{2} \tag{29}$$

4.1 Various theorems

$$1 = \sin^2 x + \cos^2 x \tag{30}$$

Addition theorems	$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \quad (31)$$ $$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \quad (32)$$ $$\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)} = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \quad (33)$$
Double angle	$$\sin 2x = 2 \sin x \cos x \quad (34)$$ $$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \quad (35)$$ $$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x} \quad (36)$$
	$$\cos x + b \sin x = \sqrt{1 + b^2} \cos(x - \theta) \quad (37)$$ $\tan \theta = b$

4.2 Table of values

Degree	0°	30°	45°	60°	90°	120°	180°	270°
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\sqrt{\pi}}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	-1
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	-1	0
$\tan(x)$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	∞	$-\sqrt{3}$	0	∞

Calculus

4.3 Convolution

Convolution is **commutative**, **associative** and **distributive**.

Definition	$$(f * g)(t) = f(t) * g(t) = int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau \tag{38}$$
Notation	$$f(t) * g(t-t_0) = (f * g)(t-t_0) \tag{39}$$ $$f(t-t_0) * g(t-t_0) = (f * g)(t-2t_0) \tag{40}$$
Commutativity	$$f * g = g * f \tag{41}$$
Associativity	$$(f * g) * h = f * (g * h) \tag{42}$$
Distributivity	$$f * (g+h) = f * g + f * h \tag{43}$$
Complex conjugate	$$(f * g)^* = f^* * g^* \tag{44}$$

4.4 Fourier analysis

4.4.1 Fourier series

Fourier series Complex representation	$$f(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{2\pi i k t}{T}\right) \tag{45}$$ $f \in \mathcal{L}^2(\mathbb{R}, \mathbb{C})$ T-periodic

Fourier coefficients
Complex representation

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\exp\left(-\frac{2\pi i}{T}kt\right)\mathrm{d}t \quad \text{for } k \geq 0 \tag{46}$$

$$c_{-k} = \overline{c_k} \quad \text{if } f \text{ real} \tag{47}$$

Fourier series
Sine and cosine representation

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{2\pi}{T}kt\right) + b_k \sin\left(\frac{2\pi}{T}kt\right)\right) \tag{48}$$

$f \in \mathcal{L}^2(\mathbb{R}, \mathbb{C})$ T-periodic

Fourier coefficients
Sine and cosine representation
If f has point symmetry: $a_{k>0} = 0$, if f has axial symmetry: $b_k = 0$

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\cos\left(-\frac{2\pi}{T}kt\right)\mathrm{d}t \quad \text{for } k \geq 0 \tag{49}$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\sin\left(-\frac{2\pi}{T}kt\right)\mathrm{d}t \quad \text{for } k \geq 1 \tag{50}$$

$$a_k = c_k + c_{-k} \quad \text{for } k \geq 0 \tag{51}$$

$$b_k = i(c_k - c_{-k}) \quad \text{for } k \geq 1 \tag{52}$$

TODO:cleanup

4.4.2 Fourier transformation

Fourier transform

$$\hat{f}(k) := \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-ikx}\, f(x)\,\mathrm{d}x \tag{53}$$

$\hat{f} : \mathbb{R}^n \mapsto \mathbb{C}$, $\forall f \in L^1(\mathbb{R}^n)$

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for $f \in L^1(\mathbb{R}^n)$:

i) $f \mapsto \hat{f}$ linear in f

ii) $g(x) = f(x-h) \quad \Rightarrow \quad \hat{g}(k) = \mathrm{e}^{-ikn}\,\hat{f}(k)$

iii) $g(x) = \mathrm{e}^{ih\cdot x}\, f(x) \quad \Rightarrow \quad \hat{g}(k) = \hat{f}(k-h)$

iv) $g(\lambda) = f\left(\frac{x}{\lambda}\right) \quad \Rightarrow \quad \hat{g}(k)\lambda^n \hat{f}(\lambda k)$

5 List of common integrals

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{(1-2^{(1-s)})\Gamma(s)}\int_0^{\infty}\mathrm{d}\eta\frac{\eta^{(s-1)}}{\mathrm{e}^{\eta}+1} \tag{54}$$

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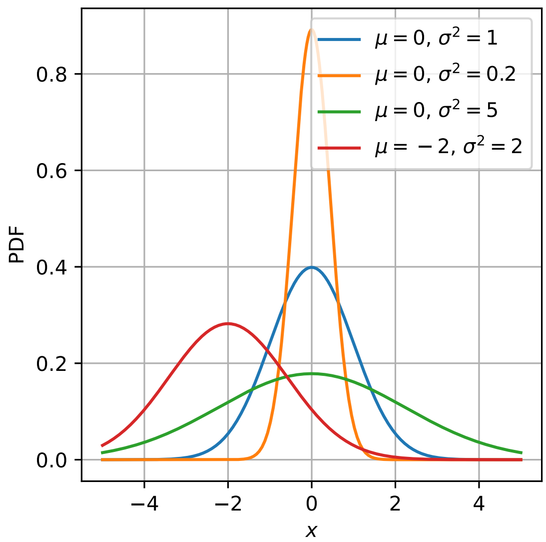
pt IV

Probability theory

Mean	$$\langle x \rangle = \int w(x)\, x \,\mathrm{d}x \tag{55}$$
Variance	$$\sigma^2 = (\Delta \hat{x})^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \langle (x - \langle x \rangle)^2 \rangle \tag{56}$$
Standard deviation	$$\sigma = \sqrt{(\Delta x)^2} \tag{57}$$
Median Value separating lower half from top half	$$\mathrm{med}(x) = \begin{cases} x_{(n+1)/2} & n \text{ odd} \\ \frac{x_{(n/2)} + x_{((n/2)+1)}}{2} & n \text{ even} \end{cases} \tag{58}$$ x dataset with n elements
Probability density function Random variable has density f. The integral gives the probability of X taking a value $x \in [a, b]$.	$$P([a,b]) := \int_a^b f(x)\,\mathrm{d}x \tag{59}$$ f normalized: $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$
Cumulative distribution function	$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t \tag{60}$$ f probability density function

6 Distributions

6.0.1 Gauß/Normal distribution

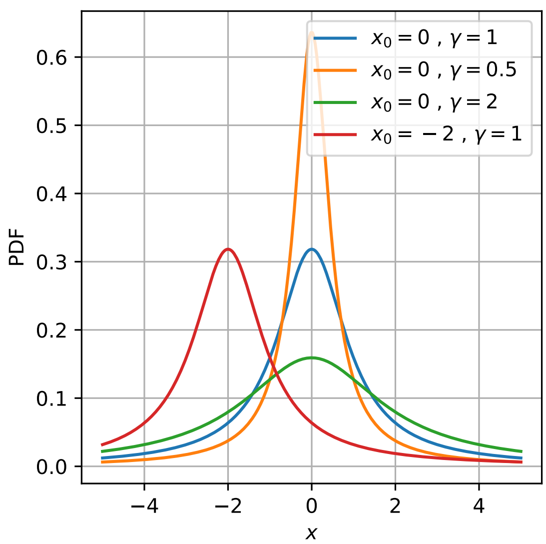


parameters	$\mu \in \mathbb{R}, \quad \sigma^2 \in \mathbb{R}$
support	$x \in \mathbb{R}$
pdf	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
cdf	$\frac{1}{2}\left[1 + \text{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$
mean	μ
median	μ
variance	σ^2

Density function of the standard normal distribution $\mu = 0, \sigma = 1$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \tag{61}$$

6.0.2 Cauchys / Lorentz distribution

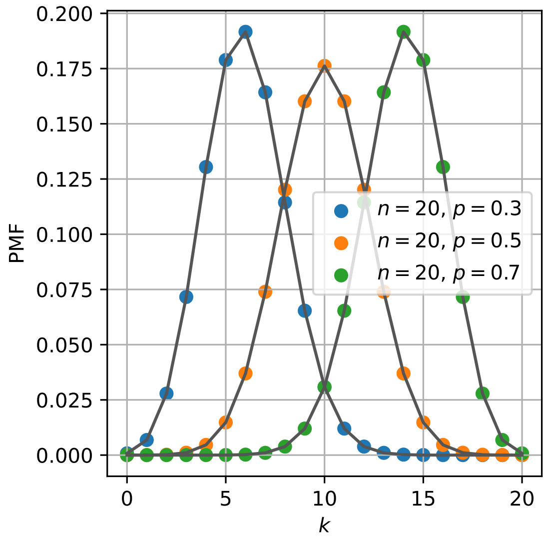


parameters	$x_0 \in \mathbb{R}, \quad \gamma \in \mathbb{R}$
support	$x \in \mathbb{R}$
pdf	$\frac{1}{\pi\gamma\left[1 + \left(\frac{x-x_0}{\gamma}\right)^2\right]}$
cdf	$\frac{1}{\pi}\arctan\left(\frac{x-x_0}{\gamma}\right) + \frac{1}{2}$
mean	undefined
median	x_0
variance	undefined

Also known as **Cauchy-Lorentz distribution**, **Lorentz(ian) function**, **Breit-Wigner distribution**.

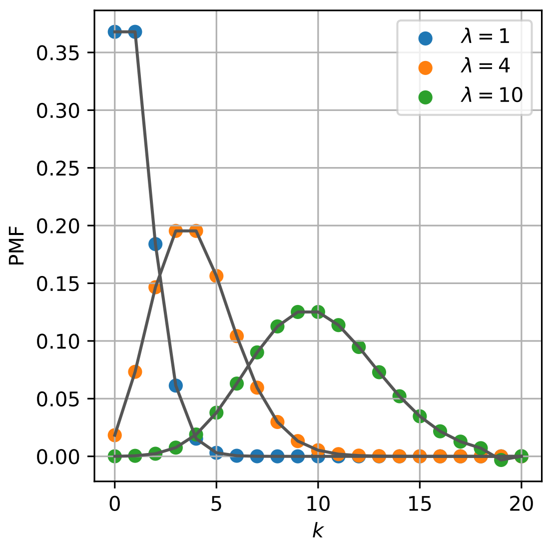
6.0.3 Binomial distribution

For the number of trials going to infinity ($n \to \infty$), the binomial distribution converges to the poisson distribution



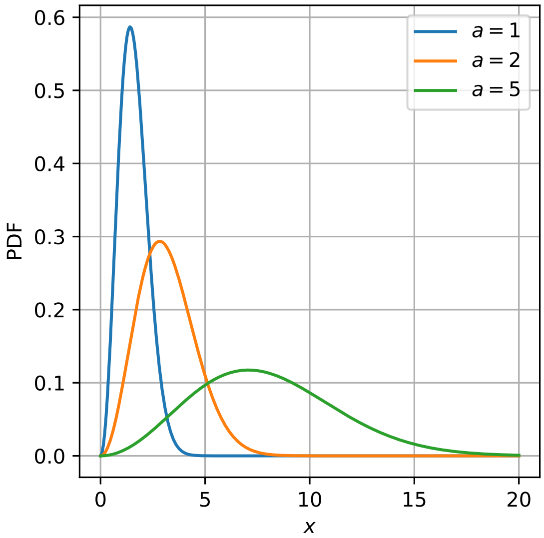
parameters	$n \in \mathbb{Z}, \quad p \in [0,1], \quad q = 1-p$
support	$k \in \{0, 1, \ldots, n\}$
pmf	$\binom{n}{k} p^k q^{n-k}$
mean	np
median	$\lfloor np \rfloor$ or $\lceil np \rceil$
variance	$npq = np(1-p)$

6.0.4 Poisson distribution



parameters	$\lambda \in (0, \infty)$
support	$k \in \mathbb{N}$
pmf	$\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!}$
cdf	$\mathrm{e}^{-\lambda} \sum_{j=0}^{\lfloor k \rfloor} \frac{\lambda^j}{j!}$
mean	λ
median	$\approx \left\lfloor \lambda + \frac{1}{3} - \frac{1}{50\lambda} \right\rfloor$
variance	λ

6.0.5 Maxwell-Boltzmann distribution



parameters	$a > 0$
support	$x \in (0, \infty)$
pdf	$\sqrt{\frac{2}{\pi}} \frac{x^2}{a^3} \exp\left(-\frac{x^2}{2a^2}\right)$
cdf	$\operatorname{erf}\left(\frac{x}{\sqrt{2}a}\right) - \sqrt{\frac{2}{\pi}} \frac{x}{a} \exp\left(\frac{-x^2}{2a^2}\right)$
mean	$2a\frac{2}{\pi}$
median	
variance	$\frac{a^2(3\pi - 8)}{\pi}$

6.1 Central limit theorem

Suppose $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with $\langle X_i \rangle = \mu$ and $(\Delta X_i)^2 = \sigma^2 < \infty$. As N approaches infinity, the random variables $\sqrt{N}(\bar{X}_N - \mu)$ converge to a normal distribution $\mathcal{N}(0, \sigma^2)$.
That means that the variance scales with $\frac{1}{\sqrt{N}}$ and statements become accurate for large N.

mech V
Mechanics

7 Lagrange formalism

The Lagrange formalism is often the most simple approach the get the equations of motion, because with suitable generalied coordinates obtaining the Lagrange function is often relatively easy.
The generalized coordinates are choosen so that the cronstraints are automatically fullfilled. For example, the generalized coordinate for a 2D pendelum is $q = \varphi$, with $\vec{x} = \begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix}$.

Lagrange function	$$\mathcal{L} = T - V \quad (62)$$ T kinetic energy, V potential energy
Lagrange equations (2nd type)	$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = 0 \quad (63)$$ q generalized coordinates
Canocial Momentum	$$p = \frac{\partial\mathcal{L}}{\partial\dot{q}} \quad (64)$$
Hamiltonian Hamiltonian can be derived from the Lagrangian using a Legendre transformation	$$H(q,p) = p\,\dot{q} - \mathcal{L}\big(q, \dot{q}(q,p)\big) \quad (65)$$

TODO:Legendre trafo

stat VI

Statistichal Mechanics

Intensive quantities: Additive for subsystems (system size dependent): $S(\lambda E, \lambda V, \lambda N) = \lambda S(E, V, N)$
Extensive quantities: Independent of system size, ratio of two intensive quantities

Liouville equation

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \{H, \rho\} \tag{66}$$

$\{\}$ poisson bracket

8 Entropy

Positive-definite and additive

$$S \geq 0 \tag{67}$$

$$S(E_1, E_2) = S_1 + S_2 \tag{68}$$

Von-Neumann

$$S = -k_B \langle \log \rho \rangle = -k_B \operatorname{tr}(\rho \log \rho) \tag{69}$$

ρ density matrix

Gibbs

$$S = -k_B \sum_n p_n \log p_n \tag{70}$$

p_n probability for micro state n

Boltzmann

$$S = k_B \log \Omega \tag{71}$$

Ω #micro states

Temperature

$$\frac{1}{T} := \left(\frac{\partial S}{\partial E} \right)_V \tag{72}$$

Pressure

$$p = T \left(\frac{\partial S}{\partial V} \right)_E \tag{73}$$

td VII
Thermodynamics

Thermal wavelength

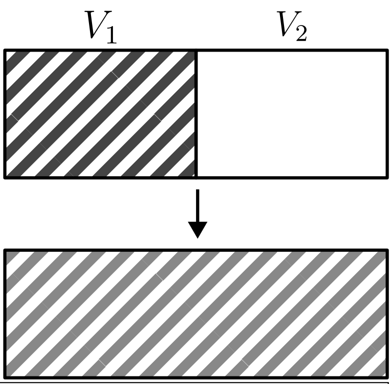
$$\lambda = \frac{\hbar}{\sqrt{2\pi m k_\mathrm{B} T}} \tag{74}$$

9 Processes

- **isobaric**: constant pressure $p = \text{const}$
- **isochoric**: constant volume $V = \text{const}$
- **isothermal**: constant temperature $T = \text{const}$
- **isentropic**: constant entropy $S = \text{const}$
- **isenthalpic**: constant entalphy $H = \text{const}$
- **adiabatic**: no heat transfer $\Delta Q = 0$
- **quasistatic**: happens so slow, the system always stays in td. equilibrium
- **reversivle**: reversible processes are always quasistatic and no entropie is created $\Delta S = 0$

9.1 Irreversible gas expansion (Gay-Lussac experiment)

A classical gas in a system with volume V_1 is separated from another system with volume V_2. In the Gay-Lussac experiment, the separation is removed and the gas flows into V_2.



Entropy change

$$\Delta S = N k_\mathrm{B} \ln\left(\frac{V_1 + V_2}{V_1}\right) > 0 \tag{75}$$

TODO:Reversible
TODO:Quasistatischer T-Ausgleich
TODO:Joule-Thompson Prozess

10 Phase transitions

A phase transition is a discontinuity in the free energy F or Gibbs energy G or in one of their derivatives. The degree of the phase transition is the degree of the derivative which exhibits the discontinuity.

Latent heat Heat required to bring substance from phase 1 to phase 2	$$Q_L = T\Delta S \quad (76)$$ ΔS entropy change of the phase transition
Clausius-Clapeyron equation Slope of the coexistence curve	$$\frac{dp}{dT} = \frac{Q_L}{T\Delta V} \quad (77)$$ ΔV Volume change of the phase transition
Phase transition At the coexistence curve	$$G_1 = G_2 \quad (78)$$ and therefore $$\mu_1 = \mu_2 \quad (79)$$
Gibbs rule / Phase rule	$$f = c - p + 2 \quad (80)$$ c #components, f #degrees of freedom, p #phases

10.0.1 Osmosis

Osmosis is the spontaneous net movement or diffusion of solvent molecules through a selectively-permeable membrane, which allows through the solvent molecules, but not the solute molecules. The direction of the diffusion is from a region of high water potential (region of lower solute concentration) to a region of low water potential (region of higher solute concentration), in the direction that tends to equalize the solute concentrations on the two sides.

Osmotic pressure	$$p_{osm} = k_B T \frac{N_c}{V} \quad (81)$$ N_c #dissolved particles

10.1 Material properties

Heat capacity	$$c = \frac{Q}{\Delta T} \quad (82)$$ Q heat
Isochoric heat capacity	$$c_v = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \quad (83)$$ U internal energy

Isobaric heat capacity	$$c_p = \left(\frac{\partial Q}{\partial T}\right)_P = \left(\frac{\partial H}{\partial T}\right)_P \quad (84)$$ H enthalpy
Bulk modules	$$K = -V\frac{\mathrm{d}p}{\mathrm{d}V} \quad (85)$$ p pressure, V initial volume
Compressibility	$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial p} \quad (86)$$
Isothermal compressibility	$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{K} \quad (87)$$
Adiabatic compressibility	$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S \quad (88)$$
Thermal expansion coefficient	$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,N} \quad (89)$$

11 Laws of thermodynamics

11.1 Zeroeth law

If two systems are each in thermal equilibrium with a third, they are also in thermal equilibrium with each other.

$$A \overset{th.eq.}{\leftrightarrow} C \quad \wedge \quad B \overset{th.eq.}{\leftrightarrow} C \quad \Rightarrow \quad A \overset{th.eq.}{\leftrightarrow} B \tag{90}$$

11.2 First law

In a process without transfer of matter, the change in internal energy, ΔU, of a thermodynamic system is equal to the energy gained as heat, Q, less the thermodynamic work, W, done by the system on its surroundings.

Internal energy change

$$\Delta U = \delta Q - \delta W \quad (91)$$
$$dU = T\,dS - p\,dV \quad (92)$$

11.3 Second law

Clausius: Heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time.
Kelvin: It is impossible for a self-acting machine, unaided by any external agency, to convey heat from one body to another at a higher temperature.

11.4 Third law

It is impussible to cool a system to absolute zero.

Entropy density

$$\lim_{T\to 0} s(T) = 0 \quad (93)$$

and therefore also

$$(94)$$
$$\lim_{T\to 0} c_V = 0 \quad (95)$$

$s = \frac{S}{N}$

12 Ensembles

Table 1: caption

	td:ensembles:mk	td:ensembles:k	td:ensembles:gk
variables	E, V, N	T, V, N	T, V, μ
partition_sum	$\Omega = \sum_n 1$	$Z = \sum_n e^{-\beta E_n}$	$Z_g = \sum_n e^{-\beta(E_n - \mu N_n)}$
probability	$p_n = \frac{1}{\Omega}$	$p_n = \frac{e^{-\beta E_n}}{Z}$	$p_n = \frac{e^{-\beta(E_n - \mu N_n}}{Z_g}$
td_pot	$S = k_B \ln \Omega$	$F = -k_B T \ln Z$	$\Phi = -k_B T \ln Z$
pressure	$p = T\left(\frac{\partial S}{\partial V}\right)_{E,N}$	$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$	$p = -\left(\frac{\partial \Phi}{\partial V}\right)_{T,\mu} = -\frac{\Phi}{V}$
entropy	$S = k_B = \ln \Omega$	$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$	$S = -\left(\frac{\partial \Phi}{\partial T}\right)_{V,\mu}$

Ergodic hypothesis
Over a long periode of time, all accessible microstates in the phase space are equiprobable

$$\langle A \rangle_{\text{Time}} = \langle A \rangle_{\text{Ensemble}} \quad (96)$$

A Observable

12.1 Potentials

Internal energy	$$\mathrm{d}U(S,V,N) = T\,\mathrm{d}S - p\,\mathrm{d}V + \mu\,\mathrm{d}N \quad (97)$$
Enthalpy	$$\mathrm{d}H(S,p,N) = T\,\mathrm{d}S + V\,\mathrm{d}p + \mu\,\mathrm{d}N \quad (98)$$
Gibbs energy	$$\mathrm{d}G(T,p,N) = -S\,\mathrm{d}T + V\,\mathrm{d}p + \mu\,\mathrm{d}N \quad (99)$$
Free energy / Helmholtz energy	$$\mathrm{d}F(T,V,N) = -S\,\mathrm{d}T - p\,\mathrm{d}V + \mu\,\mathrm{d}N \quad (100)$$
Grand canonical potential	$$\mathrm{d}\Phi(T,V,\mu) = -S\,\mathrm{d}T - p\,\mathrm{d}V - N\,\mathrm{d}\mu \quad (101)$$

TODO:Maxwell Relationen, TD Quadrat

13 Ideal gas

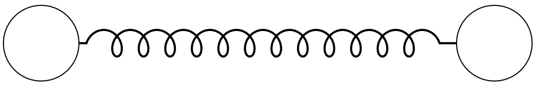
The ideal gas consists of non-interacting, undifferentiable particles.

Phase space volume $3N$ sphere	$$\Omega(E) = \int_V \mathrm{d}^3 q_1 \dots \int_V \mathrm{d}^3 q_N \int \mathrm{d}^3 p_1 \dots \int \mathrm{d}^3 p_N \frac{1}{N!\,h^{3N}} \Theta\left(E - \sum_i \frac{\vec{p}_i^{\,2}}{2m}\right) \quad (102)$$ $$= \left(\frac{V}{N}\right)^N \left(\frac{4\pi m E}{3h^2 N}\right)^{\frac{3N}{2}} \mathrm{e}^{\frac{5N}{2}} \quad (103)$$ N #particles, h^{3N} volume of a microstate, $N!$ particles are undifferentiable
Entropy	$$S = \frac{5}{2} N k_\mathrm{B} + N k_\mathrm{B} \ln\left(\frac{V}{N}\left(\frac{2\pi m E}{3h^2 N}\right)^{\frac{3}{2}}\right) \quad (104)$$

Ideal gas equation	$$pV = nRT \quad (105)$$ $$= Nk_{\mathrm{B}}T \quad (106)$$
Equation of state	$$U = \frac{3}{2}Nk_{\mathrm{B}}T \quad (107)$$
Equipartitiontheorem Each degree of freedom contributes U_{D} (for classical particle systems)	$$U_{\mathrm{D}} = \frac{1}{2}k_{\mathrm{B}}T \quad (108)$$
Maxwell velocity distribution See also 6.0.5	$$w(v)\,\mathrm{d}v = 4\pi\left(\frac{\beta m}{2\pi}\right)^{\frac{3}{2}} v^2\,\mathrm{e}^{-\frac{\beta m v^2}{2}}\,\mathrm{d}v \quad (109)$$
Average quadratic velocity per particle in a 3D gas	$$\langle v^2\rangle = \int_0^\infty \mathrm{d}v\, v^2 w(v) = \frac{3k_{\mathrm{B}}T}{m} \quad (110)$$

13.0.1 Molecule gas

Molecule gas
2 particles of mass M connected by a “spring” with distance L



Translation	$$p_i = \frac{2\pi\hbar}{L}n_i \quad (111)$$ $$E_{\mathrm{kin}} = \frac{\vec{p}_r^2}{2M} \quad (112)$$ $n_i \in \mathbb{N}_0,\ i = x, y, z$
Vibration	$$E_{\mathrm{vib}} = \hbar\omega\left(n + \frac{1}{2}\right) \quad (113)$$ $n \in \mathbb{N}_0$
Rotation	$$E_{\mathrm{rot}} = \frac{\hbar^2}{2I}j(j+1) \quad (114)$$ $j \in \mathbb{N}_0$

TODO:Diagram für verschiedene Temperaturen, Weiler Skript p.83

14 Real gas

14.1 Virial expansion

Expansion of the pressure p in a power series of the density ρ.

Virial expansion
The 2^{nd} and 3^{d} virial coefficient are tabelated for many substances

$$p = k_{\mathrm{B}} T \rho \left[1 + B(T)\rho + C(T)\rho^2 + \dots\right] \tag{115}$$

B and C 2^{nd} and 3^{d} virial coefficient, $\rho = \frac{N}{V}$

Mayer function

$$f(\vec{r}) = \mathrm{e}^{-\beta V(i,j)} - 1 \tag{116}$$

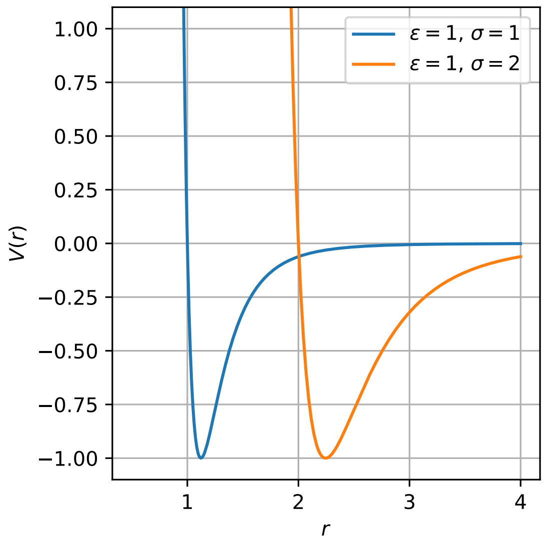
$V(i,j)$ pair potential

Second virial coefficient
Depends on pair potential between two molecules

$$B = -\frac{1}{2} \int_V \mathrm{d}^3\vec{r} f(\vec{r}) \tag{117}$$

Lennard-Jones potential
Potential between two molecules. Attractive for $r > \sigma$, repulsive for $r < \sigma$

$$V(r) = 4\epsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \tag{118}$$



14.2 Van der Waals equation

Assumes a hard-core potential with a weak attraction.

Partition sum

$$Z_N = \frac{(V - V_0)^N}{\lambda^{3N} N!} \mathrm{e}^{\frac{\beta N^2 a}{V}} \tag{119}$$

a internal pressure

Van der Waals equation	$$p = \frac{Nk_\mathrm{B}T}{V-b} - \frac{N^2 a}{V^2} \quad (120)$$ b co-volume?

TODO:sometimes N is included in a, b

15 Ideal quantum gas

Fugacity	$$z = \mathrm{e}^{\mu\beta} = \mathrm{e}^{\frac{\mu}{k_\mathrm{B}T}} \quad (121)$$
Occupation number	$$\sum_r n_r = N \quad (122)$$ r states
Undifferentiable particles	$$\lvert p_1, p_2, \dots, p_N\rangle = \lvert p_1\rangle \lvert p_2\rangle \dots \lvert p_N\rangle \quad (123)$$ p_i state
Applying the parity operator yields a *symmetric* (Bosons) and a *antisymmetic* (Fermions) solution	$$\hat{P}_{12}\psi(p_i(\vec{r}_1), p_j(\vec{r}_2)) = \pm\psi(p_i(\vec{r}_1), p_j(\vec{r}_2)) \quad (124)$$ $\hat{P}_{12}$ parity operator swaps 1 and 2, $\pm$: $\substack{\text{bos}\\\text{fer}}$
Spin degeneracy factor	$$g_s = 2s + 1 \quad (125)$$ s spin
Density of states	$$g(\epsilon) = g_s \frac{V}{4\pi}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\epsilon} \quad (126)$$ g_s td:id_qgas:spin_degeneracy_factor
Occupation number per energy	$$n(\epsilon)\,\mathrm{d}\epsilon = \frac{g(\epsilon)}{\mathrm{e}^{\beta(\epsilon-\mu)} \mp 1}\,\mathrm{d}\epsilon \quad (127)$$ td:id_qgas:dos , $\pm$: $\substack{\text{bos}\\\text{fer}}$

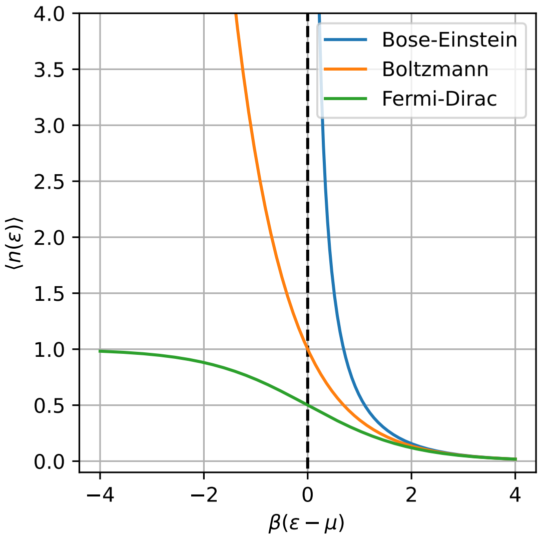
Occupation number

$$\langle n(\epsilon)\rangle = \frac{1}{e^{\beta(\epsilon-\mu)} \mp 1} \tag{128}$$

for $\epsilon - \mu \gg k_B T$

$$= \frac{1}{e^{\beta(\epsilon-\mu)}} \tag{129}$$

$\pm$: $\substack{\text{bos}\\\text{fer}}$



Number of particles

$$\langle N\rangle = \int_0^\infty n(\epsilon)\,d\epsilon \tag{130}$$

Energy
Equal to the classical ideal gas

$$\langle E\rangle = \int_0^\infty \epsilon n(\epsilon)\,d\epsilon = \frac{3}{2}pV \tag{131}$$

Equation of state
Bosons: decreased pressure, they like to cluster
Fermions: increased pressure because of the Pauli principle

$$pV = k_B T \ln Z_g \tag{132}$$

after Virial expansion

$$= N k_B T\left[1 \mp \frac{\lambda^3}{2^{5/2} g v} + \mathcal{O}\left(\left(\frac{\lambda^3}{v}\right)^2\right)\right] \tag{133}$$

$\pm$: $\substack{\text{bos}\\\text{fer}}$, $v = \frac{V}{N}$ specific volume

---

Relevance of qm. corrections
Corrections become relevant when the particle distance is in the order of the thermal wavelength

$$\left(\frac{V}{N}\right)^{\frac{1}{3}} \sim \frac{\lambda}{g_s^{\frac{1}{3}}} \tag{134}$$

Generalized zeta function

$$\left.\begin{matrix} g_\nu(z) \\ f_\nu(z) \end{matrix}\right\} := \frac{1}{\Gamma(\nu)} \int_0^\infty dx\, \frac{x^{\nu-1}}{e^x z^{-1} \mp 1} \tag{135}$$

15.1 Bosons

Partition sum

$$Z_{\mathrm{g}} = \prod_p \frac{1}{1 - \mathrm{e}^{-\beta(\epsilon_p - \mu)}} \tag{136}$$

$p \in \mathbb{N}_0$

Occupation number
Bose-Einstein distribution

$$\langle n_p \rangle = \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} - 1} \tag{137}$$

15.2 Fermions

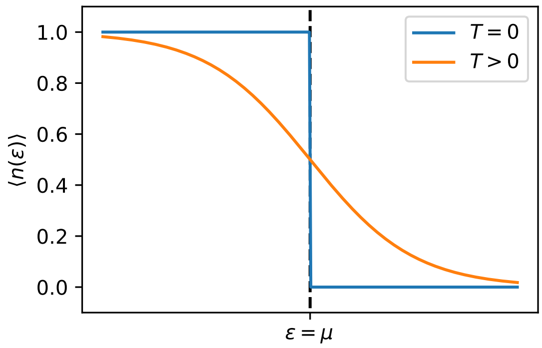
Partition sum

$$Z_{\mathrm{g}} = \prod_p \left(1 + \mathrm{e}^{-\beta(\epsilon_p - \mu)}\right) \tag{138}$$

$p = 0, 1$

Occupation number
Fermi-Dirac distribution. At $T = 0$ *Fermi edge* at $\epsilon = \mu$

$$\langle n_p \rangle = \frac{1}{\mathrm{e}^{\beta(\epsilon-\mu)} + 1} \tag{139}$$



Slater determinant

$$\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} p_1(\vec{r}_1) & p_2(\vec{r}_1) & \dots & p_N(\vec{r}_1) \\ p_1(\vec{r}_2) & p_2(\vec{r}_2) & \dots & p_N(\vec{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\vec{r}_N) & p_2(\vec{r}_N) & \dots & p_N(\vec{r}_N) \end{vmatrix} \tag{140}$$

Fermi energy

$$\epsilon_{\mathrm{F}} := \mu(T = 0) \tag{141}$$

Fermi temperature

$$T_{\mathrm{F}} := \frac{\epsilon_{\mathrm{F}}}{k_{\mathrm{B}}} \tag{142}$$

Fermi impulse Radius of the *Fermi sphere* in impulse space. States with p_F are in the *Fermi surface*	$$p_F = \hbar k_F = (2mE_F)^{\frac{1}{2}} \quad (143)$$
Specific density	$$v = \frac{N}{V} = \frac{g}{\lambda^3} f_{3/2}(z) \quad (144)$$ f td:id_qgas:generalized_zeta , g degeneracy factor, z td:id_qgas:fugacity

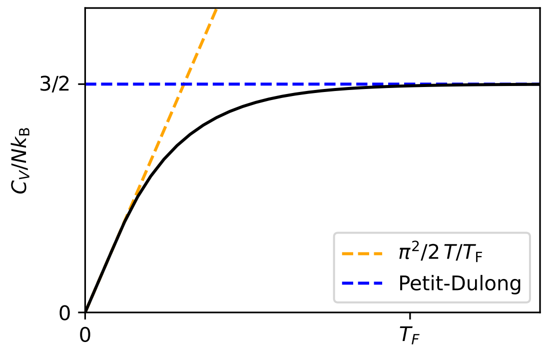
15.2.1 Strong degeneracy

Sommerfeld expansion for low temperatures $T \ll T_F$	$$f_\nu(z) = \frac{(\ln z)^\nu}{\Gamma(\nu+1)} \left(1 + \frac{\pi^6}{6} \frac{\nu(\nu-1)}{(\ln z)^2} + \dots\right) \quad (145)$$
Energy density	$$\frac{E}{V} = \frac{3}{2} \frac{g}{\lambda^3} k_B T f_{5/2}(z) \quad (146)$$ td:id_qgas:fer:degenerate:sommerfeld $$\approx \frac{3}{5} \frac{N}{V} E_F \left(1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{E_F}\right)^2\right) \quad (147)$$

Heat capacity
for low temperatures $T \ll T_F$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = N k_B \frac{\pi}{2} \left(\frac{T}{T_F}\right) \quad (148)$$

differs from td:TODO:petit_dulong



TODO:Entartung und Sommerfeld TODO:DULONG-PETIT Gesetz ed VIII

Electrodynamics

16 Maxwell-Equations

Vacuum
microscopic formulation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{\text{el}}}{\epsilon_0} \quad (149)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (150)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \quad (151)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\mathrm{d}\vec{E}}{\mathrm{d}t} \quad (152)$$

Matter
Macroscopic formulation

$$\vec{\nabla} \cdot \vec{D} = \rho_{\text{el}} \quad (153)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (154)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \quad (155)$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\mathrm{d}\vec{D}}{\mathrm{d}t} \quad (156)$$

17 Fields

17.1 Electric field

Gauss's law for electric fields
Electric flux through a closed surface is proportional to the electric charge

$$\Phi_{\text{E}} = \iint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\varepsilon_0} \quad (157)$$

S closed surface

17.2 Electric field

Magnetic flux

$$\Phi_{\text{B}} = \iint_A \vec{B} \cdot \mathrm{d}\vec{A} \quad (158)$$

Gauss's law for magnetism
Magnetic flux through a closed surface is 0 ⇒there are no magnetic monopoles

$$\Phi_{\text{B}} = \iint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0 \quad (159)$$

S closed surface

	(160)
Magnetization	$$\vec{M} = \frac{\mathrm{d}\vec{m}}{\mathrm{d}V} = \chi_\mathrm{m} \cdot \vec{H} \quad (161)$$ m mag. moment, V volume
Torque	$$\vec{\tau} = \vec{m} \times \vec{B} \quad (162)$$ m mag. moment
Susceptibility	$$\chi_\mathrm{m} = \frac{\partial M}{\partial B} = \frac{\mu}{\mu_0} - 1 \quad (163)$$
Poynting vector Directional energy flux or power flow of an electromagnetic field [W/m^2]	$$\vec{S} = \vec{E} \times \vec{H} \quad (164)$$

17.3 Induction

Faraday's law of induction	$$U_\mathrm{ind} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\mathrm{B} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_A \vec{B} \cdot \mathrm{d}\vec{A} \quad (165)$$

18 Hall-Effect

Cyclontron frequency	$$\omega_\mathrm{c} = \frac{eB}{m_\mathrm{e}} \quad (166)$$

TODO:Move

18.1 Classical Hall-Effect

Current flowing in x direction in a conductor $(l \times b \times d)$ with a magnetic field B in z direction leads to a hall voltage U_H in y direction.

Hall voltage	$$U_H = \frac{IB}{ned} \quad (167)$$ n charge carrier density
Hall coefficient	$$R_H = -\frac{Eg}{j_x B g} = \frac{1}{ne} = \frac{\rho_{xy}}{B_z} \quad (168)$$
Resistivity	$$\rho_{xx} = \frac{m_e}{ne^2\tau} \quad (169)$$ $$\rho_{xy} = \frac{B}{ne} \quad (170)$$

18.2 Integer quantum hall effect

Conductivity tensor	$$\sigma = \begin{pmatrix} \sigma_{xy} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \quad (171)$$
Resistivity tensor	$$\rho = \sigma^{-1} \quad (172)$$
Resistivity	$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \quad (173)$$ $\nu \in \mathbb{Z}$

TODO:sort

Impedance of a capacitor	$$Z_C = \frac{1}{i\omega C} \quad (174)$$
Impedance of an inductor	$$Z_L = i\omega L \quad (175)$$

TODO:impedance addition for parallel / linear

19 Dipole-stuff

Dipole radiation Poynting vector	$$\vec{S} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2} \vec{r} \quad (176)$$
Time-average power	$$P = \frac{\mu_0 \omega^4 p_0^2}{12\pi c} \quad (177)$$

qm IX

Quantum Mechanics

20 Basics

20.1 Operators

Dirac notation

$$\langle x| \text{"Bra" Row vector} \quad (178)$$
$$|x\rangle \text{"Ket" Column vector} \quad (179)$$
$$\hat{A}\,|\beta\rangle = |\alpha\rangle \Rightarrow \langle\alpha| = \langle\beta|\,\hat{A}^\dagger \quad (180)$$

Dagger

$$\hat{A}^\dagger = (\hat{A}^*)^\mathrm{T} \quad (181)$$
$$(c\hat{A})^\dagger = c^*\hat{A}^\dagger \quad (182)$$
$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \quad (183)$$
$$(184)$$

Adjoint operator

$$\langle\alpha|\hat{A}^\dagger|\beta\rangle = \langle\beta|\hat{A}|\alpha\rangle^* \quad (185)$$

Hermitian operator

$$\hat{A} = \hat{A}^\dagger \quad (186)$$

20.2 Probability theory

Continuity equation

$$\frac{\partial\rho(\vec{x},t)}{\partial t} + \nabla\cdot\vec{j}(\vec{x},t) = 0 \quad (187)$$

ρ density of a conserved quantity q, j flux density of q

State probability

$$\mathit{TODO} \quad (188)$$

Dispersion

$$\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle \quad (189)$$

Generalized uncertainty principle

$$\sigma_A \sigma_B \geq \frac{1}{4} \langle [\hat{A}, \hat{B}] \rangle^2 \tag{190}$$

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| \tag{191}$$

20.2.1 Pauli matrices

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle \langle 1| + |1\rangle \langle 0| \tag{192}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\,|0\rangle \langle 1| + i\,|1\rangle \langle 0| \tag{193}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle \langle 0| - |1\rangle \langle 1| \tag{194}$$

20.3 Commutator

Commutator

$$[A, B] = AB - BA \tag{195}$$

Anticommutator

$$\{A, B\} = AB + BA \tag{196}$$

Commutation relations

$$[A, BC] = [A, B]C - B[A, C] \tag{197}$$

TODO:add some more?

Commutator involving a function

$$[f(A), B] = [A, B] \frac{\partial f}{\partial A} \tag{198}$$

given $[A, [A, B]] = 0$

Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{199}$$

Hadamard's Lemma	$$e^A\, B\, e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \quad (200)$$
Canonical commutation relation	$$[x_i, x_j] = 0 \quad (201)$$ $$[p_i, p_j] = 0 \quad (202)$$ $$[x_i, p_j] = i\hbar\delta_{ij} \quad (203)$$ x, p canonical conjugates

21 Schrödinger equation

Energy operator	$$E = i\hbar\frac{\partial}{\partial t} \quad (204)$$
Momentum operator	$$\vec{p} = -i\hbar\vec{\nabla}_x \quad (205)$$
Space operator	$$\vec{x} = i\hbar\vec{\nabla}_p \quad (206)$$
Stationary Schrödingerequation	$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \quad (207)$$
Schrödinger equation	$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = (-\frac{\hbar^2}{2m}\vec{\nabla}^2 + \vec{V}(x))\psi(x) \quad (208)$$

21.1 Time evolution

The time evolution of the Hamiltonian is given by:

Time evolution operator	$$|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle \quad (209)$$ U unitary

Von-Neumann Equation Time evolution of the density operator in the Schrödinger picture. Qm analog to the Liouville equation ??	$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] \quad (210)$$
Lindblad master equation Generalization of von-Neummann equation for open quantum systems	$$\dot{\rho} = \underbrace{-\frac{i}{\hbar}[\hat{H}, \rho]}_{\text{reversible}} + \underbrace{\sum_{n.m} h_{nm} \left(\hat{A}_n \rho \hat{A}_{m^\dagger} - \frac{1}{2} \left\{ \hat{A}_m^\dagger \hat{A}_n, \rho \right\} \right)}_{\text{irreversible}} \quad (211)$$ h positive semidifnite matrix, $\hat{A}$ arbitrary operator

TODO:unitary transformation of time dependent H

21.1.1 Schrödinger- and Heisenberg-pictures

In the **Schrödinger picture**, the time dependecy is in the states while in the **Heisenberg picture** the observables (operators) are time dependent.

Schrödinger time evolution	$$\lvert \psi(t)_{\mathrm{S}} \rangle = \hat{U}(t, t_0) \lvert \psi(t_0) \rangle \quad (212)$$
Heisenberg time evolution	$$\lvert \psi_{\mathrm{H}} \rangle = \lvert \psi_{\mathrm{S}}(t_0) \rangle \quad (213)$$ $$A_{\mathrm{H}} = U^\dagger(t, t_0) A_{\mathrm{S}} U(t, t_0) \quad (214)$$ $$\frac{\mathrm{d}\hat{A}_{\mathrm{H}}}{\mathrm{d}t} = \frac{1}{i\hbar}[\hat{A}_{\mathrm{H}}, \hat{H}_{\mathrm{H}}] + \left(\frac{\partial \hat{A}_{\mathrm{S}}}{\partial t} \right)_{\mathrm{H}} \quad (215)$$ H and S being the Heisenberg and Schrödinger picture, respectively

21.1.2 Ehrenfest theorem

See also ??

Ehrenfesttheorem applies to both pictures	$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{A} \rangle = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle \quad (216)$$
Example for x	$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle x \rangle = -\langle \nabla V(x) \rangle = \langle F(x) \rangle \quad (217)$$

21.2 Correspondence principle

The classical mechanics can be derived from quantum mechanics in the limit of large quantum numbers.

22 Pertubation theory

qm:qm_pertubation:desc

Hamiltonian	$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1 \quad (218)$$
Power series	$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots \quad (219)$$ $$\lvert\psi_n\rangle = \lvert\psi_n^{(0)}\rangle + \lambda \lvert\psi_n^{(1)}\rangle + \lambda^2 \lvert\psi_n^{(2)}\rangle + \ldots \quad (220)$$
1. order energy shift	$$E_n^{(1)} = \left\langle \psi_n^{(0)} \middle\vert \hat{H}_1 \middle\vert \psi_n^{(0)} \right\rangle \quad (221)$$
1. order states	$$\lvert\psi_n^{(1)}\rangle = \sum_{k\neq n} \frac{\left\langle \psi_k^{(0)} \middle\vert \hat{H}_1 \middle\vert \psi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}} \lvert\psi_k^{(0)}\rangle \quad (222)$$
2. order energy shift	$$E_n^{(2)} = \sum_{k\neq n} \frac{\left\lvert \left\langle \psi_k^{(0)} \middle\vert \hat{H}_1 \middle\vert \psi_n^{(0)} \right\rangle \right\rvert^2}{E_n^{(0)} - E_k^{(0)}} \quad (223)$$
Fermiś golden rule Transition rate from initial state $\lvert i\rangle$ under a pertubation H^1 to final state $\lvert f\rangle$	$$\Gamma_{i\to f} = \frac{2\pi}{\hbar} \left\lvert \langle f\vert H^1 \vert i\rangle \right\rvert^2 \rho(E_f) \quad (224)$$

23 Harmonic oscillator

Hamiltonian	$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad (225)$$ $$= \frac{1}{2}\hbar\omega + \omega a^\dagger a \quad (226)$$
Energy spectrum	$$E_n = \hbar\omega\left(\frac{1}{2} + n\right) \quad (227)$$

See also ??

23.1 Creation and Annihilation operators / Ladder operators

Particle number operator/occupation number operator

$$\hat{N} := a^\dagger a \tag{228}$$
$$\hat{N}\,|n\rangle = n\,|N\rangle \tag{229}$$

$|n\rangle$ = Fock states, $\hat{a}$ = Annihilation operator, $\hat{a}^\dagger$ = Creation operator

Commutator

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{230}$$
$$[N, \hat{a}] = -\hat{a} \tag{231}$$
$$[N, \hat{a}^\dagger] = \hat{a}^\dagger \tag{232}$$

Application on states

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{233}$$
$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{234}$$
$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0\rangle \tag{235}$$

23.1.1 Harmonischer Oszillator

Harmonic oscillator

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \tag{236}$$
$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger) \tag{237}$$
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{238}$$
$$a = \frac{1}{\sqrt{2}}(\tilde{X} + i\tilde{P}) \tag{239}$$
$$a^\dagger = \frac{1}{\sqrt{2}}(\tilde{X} - i\tilde{P}) \tag{240}$$

24 Angular momentum

Bloch waves
Solve the stat. SG in periodic potential with period $\vec{R}$:
$V(\vec{r}) = V(\vec{r} + \vec{R})$

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \cdot u_{\vec{k}}(\vec{r}) \tag{241}$$

$\vec{k}$ arbitrary, u periodic function

24.1 Aharanov-Bohm effect

Acquired phase
Electron along a closed loop aquires a phase proportional to the enclosed magnetic flux

$$\delta = \frac{2e}{\hbar} \oint \vec{A} \cdot \mathrm{d}\vec{s} = \frac{2e}{\hbar} \Phi \tag{242}$$

TODO:replace with loop intergral symbol and add more info

25 Symmetries

Most symmetry operators are unitary ?? because the norm of a state must be invariant under transformations of space, time and spin.

Invariance
$\hat{H}$ is invariant under a symmetrie described by $\hat{U}$ if this holds

$$\hat{U}\hat{H}\hat{U}^\dagger = \hat{H} \Leftrightarrow [\hat{U}, \hat{H}] = 0 \tag{243}$$

25.1 Time-reversal symmetry

Time-reversal symmetry

$$T : t \to -t \tag{244}$$

Anti-unitary

$$T^2 = -1 \tag{245}$$

26 Two-level systems (TLS)

James-Cummings Hamiltonian
TLS interacting with optical cavity

$$H = \underbrace{\hbar\omega_c \hat{a}^\dagger \hat{a}}_{\text{field}} + \underbrace{\hbar\omega_a \frac{\hat{\sigma}_z}{2}}_{\text{atom}} + \underbrace{\frac{\hbar\Omega}{2}\hat{E}\hat{S}}_{\text{int}} \tag{246}$$

after RWA:

$$\tag{247}$$

$$= \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_a \hat{\sigma}^\dagger \hat{\sigma} + \frac{\hbar\Omega}{2}(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger \hat{\sigma}) \tag{248}$$

$\hat{E} = E_{\text{ZPF}}(\hat{a} + \hat{a}^\dagger)$ field operator with bosonic ladder operators, $\hat{S} = \hat{\sigma}^\dagger + \hat{\sigma}$ polarization operator with ladder operators of the TLS

27 Other

Rotating Wave Approximation (RWS)
Rapidly oscilating terms are neglected

$$\Delta\omega := |\omega_0 - \omega_\mathrm{L}| \ll |\omega_0 + \omega_\mathrm{L}| \approx 2\omega_0 \tag{249}$$

ω_L light frequency, ω_0 transition frequency

28 Hydrogen Atom

Reduced mass

$$\mu = \frac{m_\mathrm{e} m_\mathrm{K}}{m_\mathrm{e} + m_\mathrm{K}} \overset{m_\mathrm{e} \ll m_\mathrm{K}}{\overset{\downarrow}{\approx}} m_\mathrm{e} \tag{250}$$

Coulumb potential
For a single electron atom

$$V(\vec{r}) = \frac{Z e^2}{4\pi\epsilon_0 r} \tag{251}$$

Z atomic number

Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2\mu}\vec{\nabla}_{\vec{r}}^2 - V(\vec{r}) \tag{252}$$

$$= \frac{\hat{p}_r^2}{2\mu} + \frac{\hat{L}^2}{2\mu r} + V(r) \tag{253}$$

Wave function

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi) \tag{254}$$

Radial part

$$R_{nl} = -\sqrt{\frac{(n-l-1)!(2\kappa)^3}{2n[(n+l)!]^3}}(2\kappa r)^l \,\mathrm{e}^{-\kappa r}\, L_{n+1}^{2l+1}(2\kappa r) \tag{255}$$

with

$$\kappa = \frac{\sqrt{2\mu|E|}}{\hbar} = \frac{Z}{n a_\mathrm{B}} \tag{256}$$

$L_r^s(x)$ Laguerre-polynomials

Energy eigenvalues

$$E_n = \frac{Z^2 \mu e^4}{n^2 (4\pi\epsilon_0)^2 2\hbar^2} = -E_\mathrm{H}\frac{Z^2}{n^2} \tag{257}$$

Rydberg energy	$$E_{\mathrm{H}} = h\,c\,R_{\mathrm{H}} = \frac{\mu e^4}{(4\pi\epsilon_0)^2 2\hbar^2} \quad (258)$$

28.1 Corrections

28.1.1 Darwin term

Relativisitc correction: Because of the electrons zitterbewegung, it is not entirely localised. TODO:fact check

Energy shift	$$\Delta E_{\mathrm{rel}} = -E_n \frac{Z^2\alpha^2}{n}\left(\frac{3}{4n} - \frac{1}{l+\frac{1}{2}}\right) \quad (259)$$
Fine-structure constant Sommerfeld constant	$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \quad (260)$$

28.1.2 Spin-orbit coupling (LS-coupling)

The interaction of the electron spin with the electrostatic field of the nuclei lead to energy shifts.

Energy shift	$$\Delta E_{\mathrm{LS}} = \frac{\mu_0 Z e^2}{8\pi m_{\mathrm{e}}^2 r^3}\langle\vec{S}\cdot\vec{L}\rangle \quad (261)$$
TODO:name	$$\begin{aligned}\langle\vec{S}\cdot\vec{L}\rangle &= \frac{1}{2}\langle[J^2 - L^2 - S^2]\rangle \\ &= \frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]\end{aligned} \quad (262)$$

28.1.3 Fine-structure

The fine-structure combines relativistic corrections 28.1.1 and the spin-orbit coupling 28.1.2.

Energy shift	$$\Delta E_{\mathrm{FS}} = \frac{Z^2\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \quad (263)$$

28.1.4 Lamb-shift

The interaction of the electron with virtual photons emitted/absorbed by the nucleus leads to a (very small) shift in the energy level.

Potential energy	$$\langle E_{\text{pot}} \rangle = -\frac{Ze^2}{4\pi\epsilon_0} \left\langle \frac{1}{r + \delta r} \right\rangle \quad (264)$$ δr pertubation of r

28.1.5 Hyperfine structure

Interaction of the nucleus spin with the magnetic field created by the electron leads to energy shifts. (Lifts degeneracy)

Nuclear spin	$$\vec{F} = \vec{J} + \vec{I} \quad (265)$$ $$\lvert\vec{I}\rvert = \sqrt{i(i+1)}\hbar \quad (266)$$ $$I_z = m_i\hbar \quad (267)$$ $$m_i = -i, -i+1, \dots, i-1, i \quad (268)$$
Combined angular momentum	$$\vec{F} = \vec{J} + \vec{I} \quad (269)$$ $$\lvert\vec{F}\rvert = \sqrt{f(f+1)}\hbar \quad (270)$$ $$F_z = m_f\hbar \quad (271)$$
Selection rule	$$f = j \pm i \quad (272)$$ $$m_f = -f, -f+1, \dots, f-1, f \quad (273)$$
Hyperfine structure constant	$$A = \frac{g_i \mu_{\text{K}} B_{\text{HFS}}}{\sqrt{j(j+1)}} \quad (274)$$ B_{HFS} hyperfine field, μ_{K} nuclear magneton, g_i nuclear g-factor ??
Energy shift	$$\Delta H_{\text{HFS}} = \frac{A}{2}[f(f+1) - j(j+1) - i(i+1)] \quad (275)$$

TODO:landé factor

28.2 Effects in magnetic field

TODO:all

TODO:Hunds rules

cm X

Condensed matter physics

29 Bravais lattice

Table 2: In 2D, there are 5 different Bravais lattices

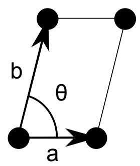
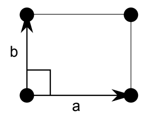
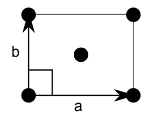
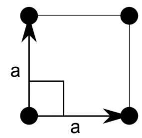
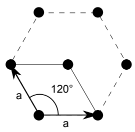
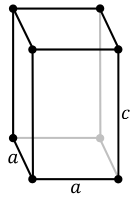
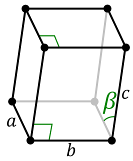
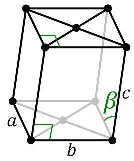
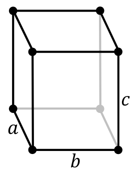
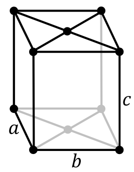
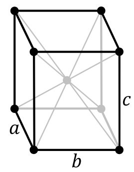
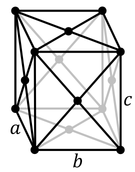
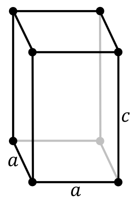
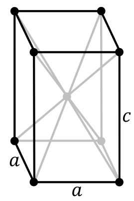
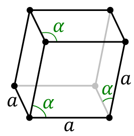
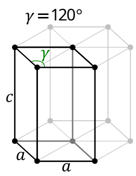
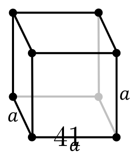
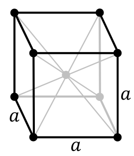
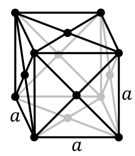
Lattice system	Point group	5 Bravais lattices	
		primitive (p)	centered (c)
monoclinic (m)	C_2		
orthorhombic (o)	D_2		
tetragonal (t)	D_4		
hexagonal (h)	D_6		

Table 3: In 3D, there are 14 different Bravais lattices

Crystal system	Lattice system	Point group	14 Bravais lattices			
			primitive (P)	base_centered (S)	body_centered (I)	face_centered (F)
triclinic (a)		C_i				
monoclinic (m)		C_{2h}				
orthorhombic (o)		D_{2h}				
tetragonal (t)		D_{4h}				
hexagonal (h)	rhombohedral	D_{3d}				
	hexagonal	D_{6h}				
cubic (c)		O_h				

30 Reciprocal lattice

The reciprokal lattice is made up of all the wave vectors $\vec{k}$ that ressemble standing waves with the periodicity of the Bravais lattice.

Reciprocal lattice vectors

$$\vec{b}_1 = \frac{2\pi}{V_c}\vec{a}_2 \times \vec{a}_3 \quad (276)$$

$$\vec{b}_2 = \frac{2\pi}{V_c}\vec{a}_3 \times \vec{a}_1 \quad (277)$$

$$\vec{b}_3 = \frac{2\pi}{V_c}\vec{a}_1 \times \vec{a}_2 \quad (278)$$

a_i real-space lattice vectors, V_c volume of the primitive lattice cell

---

30.1 Scattering processes

Matthiessen's rule
Approximation, only holds if the processes are independent of each other

$$\frac{1}{\mu} = \sum_{i=\text{Scattering processes}} \frac{1}{\mu_i} \quad (279)$$

$$\frac{1}{\tau} = \sum_{i=\text{Scattering processes}} \frac{1}{\tau_i} \quad (280)$$

μ mobility, τ scattering time

---

31 Free electron gas

Assumptions: electrons can move freely and independent of each other.

Drift velocity
Velocity component induced by an external force (eg. electric field)

$$\vec{v}_\text{D} = \vec{v} - \vec{v}_\text{th} \quad (281)$$

v_th thermal velocity

Mean free time

$$\tau \quad (282)$$

Mean free path

$$\ell = \langle v \rangle \tau \quad (283)$$

Electrical mobility

$$\mu = \frac{q\tau}{m} \quad (284)$$

q charge, m mass

31.1 Drude model

Classical model describing the transport properties of electrons in materials (metals): The material is assumed to be an ion lattice and with freely moving electrons (electron gas). The electrons are accelerated by an electric field and decelerated through collisions with the lattice ions. The model disregards the Fermi-Dirac partition of the conducting electrons.

Equation of motion

$$m_{\mathrm{e}} \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{m_{\mathrm{e}}}{\tau} \vec{v}_{\mathrm{D}} = -e\vec{E} \tag{285}$$

v electron speed, $\vec{v}_{\mathrm{D}}$ drift velocity, τ mean free time between collisions

Current density
Ohm's law

$$\vec{j} = -ne\vec{v}_{\mathrm{D}} = ne\mu\vec{E} \tag{286}$$

n charge particle density

Drude-conductivity

$$\sigma = \frac{\vec{j}}{\vec{E}} = \frac{e^2 \tau n}{m_{\mathrm{e}}} = ne\mu \tag{287}$$

31.2 Sommerfeld model

Assumes a gas of free fermions underlying the pauli-exclusion principle. Only electrons in an energy range of $k_{\mathrm{B}}T$ around the Fermi energy E_{F} participate in scattering processes.

Current density

$$\vec{j} = -en \langle v \rangle = -en \frac{\hbar}{m_{\mathrm{e}}} \langle \vec{k} \rangle = -e \frac{1}{V} \sum_{\vec{k},\sigma} \frac{\hbar \vec{k}}{m_{\mathrm{e}}} \tag{288}$$

TODO:The formula for the conductivity is the same as in the drude model?

31.3 2D electron gas

Lower dimension gases can be obtained by restricting a 3D gas with infinetly high potential walls on a narrow area with the width L.

Confinement energy
Raises ground state energy

$$\Delta E = \frac{\hbar^2 \pi^2}{2m_{\mathrm{e}} L^2} \tag{289}$$

Energy

$$E_n = \underbrace{\frac{\hbar^2 k_\parallel^2}{2m_{\mathrm{e}}}}_{x\text{-}y\text{: plain wave}} + \underbrace{\frac{\hbar^2 \pi^2}{2m_{\mathrm{e}} L^2} n^2}_{z} \tag{290}$$

31.4 1D electron gas / quantum wire

Energy	$$E_n = \frac{\hbar^2 k_x^2}{2m_e} + \frac{\hbar^2\pi^2}{2m_e L_z^2} n_1^2 + \frac{\hbar^2\pi^2}{2m_e L_y^2} n_2^2 \quad (291)$$

31.5 0D electron gas / quantum dot

TODO:TODO

32 Measurement techniques

32.1 ARPES

what? in? how? plot

32.2 Scanning probe microscopy SPM

Images of surfaces are taken by scanning the specimen with a physical probe.

Name	Atomic force microscopy (AMF)
Application	Surface stuff
how	With needle

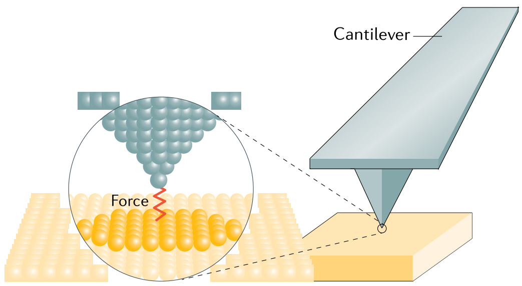


Figure 1: [?]

Name	Scanning tunneling microscopy (STM)
Application	Surface stuff
how	With TUnnel

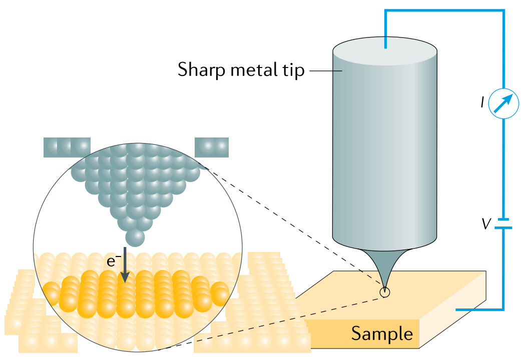
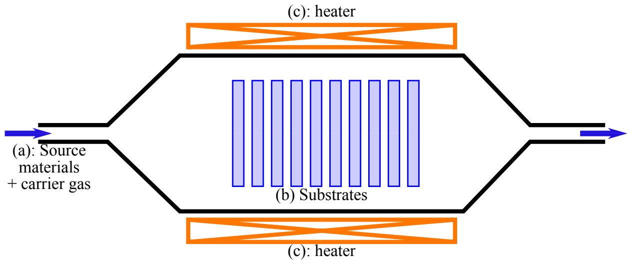


Figure 2: [?]

33 Fabrication techniques

Name	Chemical vapor deposition (CVD)
how	A substrate is exposed to volatile precursors, which react and/or decompose on the heated substrate surface to produce the desired deposit. By-products are removed by gas flow through the chamber.
Application	• Polysilicon Si • Silicon dioxide SiO_2 • Graphene • Diamond



33.1 Epitaxy

A type of crystal groth in which new layers are formed with well-defined orientations with respect to the crystalline seed layer.

Name	Molecular Beam Epitaxy (MBE)
how	In a ultra-high vacuum, the elements are heated until they slowly sublime. The gases then condensate on the substrate surface
Application	• Gallium arsenide GaAs TODO:Link to GaAs